

Asymptotic Optimality of Multiple Hypothesis Testing Procedure in Equi-correlated Set-up

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Abstract: In this paper, we have used subset selection approach to select the significant hypotheses in the context of multiple hypothesis testing problem, stated in [1] **Bogdan, Chakrabarty, Frommlet and Ghosh(2011)**. The above paper considers the problem of identifying the hypotheses which correspond to greater variance of normal variate in an independent set up, and they have considered the problem for dependent normal variables (in particular for equi-correlated set up) to be an open problem. We have explored the problem both theoretically and through extensive simulations. We have found asymptotically optimal procedure for the equi-correlated case. Though in the above mentioned paper the results were done in the context of sparsity, our results are considered under sparsity and also for the general case.

Key words and phrases: Asymptotically Optimal Selection Procedure, Multiple Hypothesis Testing, Prior, Subset Selection.

1 Introduction

In recent statistical inference problem, multiple testing procedure has emerged as an extremely important phenomena. Over the years various procedure has been suggested in this literature depending on the objective. For example we can state about the Bonferroni Procedure which controls the Family Wise Error Rate(FWER), [3] **Benjamini-Hochberg** Procedure which tries to control the False Discovery Rate(FDR).

In our case we have developed a method to solve this problem when the hypotheses are not independent of each other. Mainly we have worked on the equi-correlated case but our method works for general case also i.e. when the set up is neither independent nor equi-correlated. Independent case was performed by [1] **Bogdan, Chakrabarty, Frommlet and Ghosh(2011)** in which the dependent structured set up was mentioned as a difficult problem. In this paper, they defined Bayes oracle in order to address the solution in the independent case model in the context of multiple hypothesis testing, and provided conditions under which the [3] **Benjamini and Hochberg (1995)** and Bonferroni procedures attain the risk of the Bayes oracle. [2] **Dutta and Ghosh (2013)** proved a similar result for the horseshoe decision rule. Our proposed procedure for the dependent case has the risk function at least as good as the risk of the methods for the independent case.

1.1 Statistical Model and The description of the Problem

Suppose we have m equi-correlated observations X_1, X_2, \dots, X_m with an assumption that $\underline{X}|\underline{\mu}$ follows $N(\underline{\mu}, \sigma_\epsilon^2 \Sigma_1)$. Here $\underline{\mu}$ represents the effect under investigation and $\sigma_\epsilon^2 \Sigma_1$ represents the variance of random noise (e.g. measurement error) where it is considered that Σ_1 is a symmetric matrix with ρ_1 's in off diagonal and 1's in diagonal. Assumption is that $\underline{\mu}$ is a random variable with distributions determined by m unobservable Bernoulli(p) random variables v_i , for some $p \in (0, 1)$. The i -th null hypothesis is $H_{0i} : v_i = 0$ with the corresponding alternative $H_{Ai} : v_i = 1$. Under H_{0i} , $\mu_i \sim N(0, \sigma_0^2)$ distribution (where $\sigma_0^2 \geq 0$) while under H_{Ai} it is assumed to follow a $N(0, \sigma_0^2 + \tau^2)$ distribution (where $\tau > 0$). In this case we assume that v_i 's are independent

but μ_i 's are not, instead they are equi-correlated which is evident from the dispersion matrix. Now $\underline{v}_0 = (v_{01}, v_{02}, \dots, v_{0m})'$ is an m -dimensional vector consisting of zeroes and ones only. Hence given $\underline{v} = \underline{v}_0$ distribution of $\underline{\mu}$ is the following:

$$\underline{\mu} | (\underline{v} = \underline{v}_0) \sim N(0, D_{v_0} \Sigma_2 D_{v_0})$$

where we define p_{v_0} as the probability of \underline{v} taking the value \underline{v}_0 i.e. $p_{v_0} = p^{||v_0||}(1-p)^{m-||v_0||}$ with $||v_0||$ = number of 1's in \underline{v}_0 vector and D_{v_0} is a diagonal matrix with $(D_{v_0})_{ii} = \sigma_0$ if $v_{0i} = 0$ and $\sqrt{(\sigma_0^2 + \tau^2)}$ if $v_{0i} = 1$. Σ_2 is also a symmetric matrix with ρ_2 's in the off diagonal and 1's in the diagonal entries. Our problem is to identify the signals i.e. identify the i 's for which the corresponding v_i 's were 1.

From the informations stated above we find the marginal distribution of \underline{X} as follows:

$$\underline{X} \sim \sum_{v_0} p_{v_0} N(0, \sigma_\varepsilon^2 \Sigma_1 + D_{v_0} \Sigma_2 D_{v_0}) \quad (1)$$

1.2 The Bayes Risk

Now in this situation we need to find the optimal test to identify the signals properly. The approach is to determine the risk function and minimising it to get the required rule. In the later part we will try to extend this situation to the asymptotic framework. In order to get the optimal rule we first need to find out the loss function which is the following :

Hypothesis	H_0 accepted	H_0 rejected
H_0 true	0	δ_0
H_0 false	δ_A	0

$$L(\underline{v}, \underline{v}^*) = \sum_{i=1}^m \delta_i (v_i - v_i^*)^2$$

where \underline{v}^* denote the original value of the 0-1 vector and \underline{v} represents the estimated value of the vector obtained by the outcome of the multiple hypothesis testing procedure. Hence $\delta_i = \delta_0$ if $v_i^* = 0$ but $v_i = 1$, and $\delta_i = \delta_A$ for the opposite set up. The risk function is defined as the expected loss which is $R(\underline{v}, \underline{v}^*) = E[L(\underline{v}, \underline{v}^*)]$. Little computations show that the expression for risk is the following:

$$\begin{aligned} R(\underline{v}, \underline{v}^*) &= EE[L(\underline{v}, \underline{v}^*) | \underline{v} = \underline{v}_0] \\ &= \sum_{i=1}^m [\delta_0(1-p)t_{1i} + \delta_A p t_{2i}] \end{aligned} \quad (2)$$

Here we are considering tests which are invariant with respect to permutations which in turn implies that $t_{1i} = t_{11}$ and $t_{2i} = t_{21} \forall i$. This is because parameter space and also marginal distribution and conditional distribution of $(X_1, X_2, \dots, X_m)^T$ remains invariant with respect to permutations. Therefore the risk takes the expression of

$$R(\underline{v}, \underline{v}^*) = m [\delta_0(1-p)t_{11} + \delta_A p t_{21}]$$

So we are to minimise $[\delta_0(1-p)t_{11} + \delta_A p t_{21}]$ among permutations invariant tests.

2 Preliminary Analysis

In order to identify the signals, one naive approach is to minimise the risk function, to find the appropriate asymptotic risk and according to the findings design the optimal test statistic which is nothing but the Neyman-Pearson Likelihood statistic. But that approach obviously cannot produce any desired output here. The reason behind that is while computing the asymptotic risk we tried to find the quantity $\lim_{m \rightarrow \infty} R_m^{\frac{1}{m}}$ which turns out to be equal to:

$$\lim_{m \rightarrow \infty} R_m^{\frac{1}{m}} = \max \left[\lim_{m \rightarrow \infty} t_{11}^{\frac{1}{m}}, \lim_{m \rightarrow \infty} t_{21}^{\frac{1}{m}} \right]$$

Some simple analysis explains why the expression of the limit of risk turns out to be the quantity described here. But these quantities are extremely difficult to manipulate even if we try to approximate them with suitable terms to solve the problem. Therefore under this circumstances we got convinced about the difficulty with this strategy and tried a completely different approach to fathom the problem.

Now for all practical purposes σ_ϵ (error in measurement of X) is small compared to the variances of μ , σ_0^2 and $\sigma_0^2 + \tau^2$. For this reason we consider $\sigma_\epsilon \approx 0$ and consider the marginal distribution of X as the following: (from 1)

$$X \sim \sum_{v_0} p_{v_0} N(0, D_{v_0} \Sigma_2 D_{v_0})$$

We also find the similar assumption in [2] Dutta and Ghosh(2013) and [4] Ghosh, Tang, Ghosh and Chakraborty(2015).

Here in our problem we have to identify the coordinates of v for which $v_i = 1$. Now the observations are equicorrelated and coming from a normal distribution with same mean but different variance. The idea is to find the integers between 1 to m for which the observed data points indicate higher variance. If we can find the observations which indicate high measure of dispersion, we can easily say that the corresponding hypothesis is false and reject that. The inference problem is then to select i_1, i_2, \dots, i_k for which $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ are 1. As we have described earlier $(X_1, X_2, \dots, X_m)'$ follows $MVN(0, D_{v_0} \Sigma_2 D_{v_0})$ which is equicorrelated as Σ_2 are equicorrelated. Therefore in order to get a measure for dispersion we try to use the quantity $\underline{Z} = (Z_1, Z_2, \dots, Z_m) = (X_1^2, X_2^2, \dots, X_m^2)$ given v_0 . Now this idea follows from the following lemma.

Lemma 1 (a) Let $(X_1, X_2, \dots, X_m)'$ follows Multivariate Normal distribution with correlation matrix \mathcal{R} and their variances are $\sigma_1^2, \sigma_2^2, \dots, \sigma_m^2$ respectively then $X_i^2 \leq_{st} X_j^2$ if and only if $\sigma_i^2 \leq \sigma_j^2$.

(b) Under the assumption of part (a), with equi-correlated correlation matrix \mathcal{R} , $X_i^2|Z \leq_{st} X_j^2|Z$ if and only if $\sigma_i^2 \leq \sigma_j^2$ where Z is a subset of $\{X_1, X_2, \dots, X_m\}$ deleted by X_i and X_j .

So from the lemma we can say that to identify the signals which are actually coming from larger variance distribution we can simply select the highest K many order statistics of \underline{Z} . Now the problem is to determine K . Practically we have no information about K according to the problem. Depending on the situation we may specify K as a random variable which determine the expected number of false positives and expected number of false negatives. Also there is a choice for constant K .

Remark 1 Last thing we have to do is to choose the random variable K where K is the number of highest order statistic of X_i^2 's to adjust expected number of false positives and expected number of false negatives. Now in this situation K may be of the following 3 kinds:

- (a) K may be of constant proportion of m .
- (b) K may be number of squared statistics which are greater than C , for some C .
- (c) Here we are clustering the observations depending on variance which implies that K can be obtained simply by clustering the observations in two parts.

3 Discussion

In the previous section we have discussed the approach of solution of the problem and which in turn boils down to selecting those observations with highest measure of dispersion (here X_i^2). Now how many observations is to select is determined by K and we need to choose K suitably. We have discussed three possibilities to be considered while considering K .

3.1 Case I

The idea is that in an asymptotic framework K and m will both depend on t the sequence through which parameters vary. Denoting them by K_t and m_t respectively, let us consider $\frac{K_t}{m_t} = \alpha_t$. From the set-up of the problem it is evident that this α_t is a measure of p_t , (if we know K). If this process is performed under sparsity i.e. $p_t \rightarrow 0$, the quantity α_t should behave like p_t i.e. even if $m_t \rightarrow \infty$, α_t cannot be a constant sequence. Furthermore α_t should be decreasing sequence going to zero. Now the rate of convergence of $\alpha_t \rightarrow 0$ should be same as that of p_t which sometime we assume to be $\frac{1}{m_t^\beta}$ where β can be between 0 and 1 or may be equal to 1 also.

3.2 Case II

The second case is when we consider K as a random variable i.e K_t may be viewed as a Binomial random variable with success probability p_t and number of observations m_t . But if in our assumption $m_t \rightarrow \infty$ and $p_t \rightarrow 0$ the event of getting one and obtaining corresponding normal observation from the population with larger variance becomes gradually more and more rare. So K_t may be considered as a Poisson random variable with parameter $(m_t p_t)$. This an example of the possible distribution of K . But In order to solve the problem we need to find a C which will be used to separate the two groups.

Now here we want to find out C by approximating K_t with normal distribution. Let

$$\begin{aligned} 1 - \alpha &= P \left(\left[\frac{K_t - m_t p_t}{\sqrt{m_t p_t}} \right] \leq z_\alpha \right) \\ &= P \left(K_t \leq u_{t,\alpha} \right) \quad (z_\alpha \text{ is } (1 - \alpha) - th \text{ quantile of standard normal variate}) \end{aligned}$$

Where $\{K_t \leq u_{t,\alpha}\}$ is the same event as $\{X_{[u_{t,\alpha}]}\} \geq C$. This C is a function of $(\tau_t^2, \sigma_t^2, \rho_t, p_t, \alpha)$

Here we can apply the same method considered above for $\rho_t \neq 0$ as for the case $\rho_t = 0$ and compare our method to the other methods.

But this way of identifying signals does not perform very good in practice because as we see by simulation, the rejection region in this set up is not suitable which leads to an enormous number of misclassification. This is not shown in the table.

3.3 Case III

C should be such that it is capable of distinguishing clusters of $N(0, \sigma_0^2 + \tau^2)$ and $N(0, \sigma_0^2)$. It can be shown that the two clusters separated by C which minimises the distance of the two group means is for $C = \frac{\sum_{i=1}^n |Y_i|^m}{n}$ if the data is $(|Y_1|^m, |Y_2|^m, \dots, |Y_n|^m)$. This follows from the following result.

Result 1 For Z_1, Z_2, \dots, Z_m , m positive observations, let \bar{Z}_1 and \bar{Z}_2 denote the mean of the two groups given by $\{Z_i : Z_i \leq C\}$ and $\{Z_i : Z_i > C\}$. An optimal C for which $\left\{ \frac{m_1 m_2}{m^2} |\bar{Z}_1 - \bar{Z}_2| \right\}$ is maximised is given by $C = \frac{1}{m} \sum_{i=1}^m Z_i$, where m_1 and m_2 are number of observations in the two groups respectively.

Remark 2 In the previous case we have observed an equivalence between K and C . This is because in our procedure C actually plays the role of a threshold for dividing the data into two parts which is precisely the role of K also.

4 Selection Process of K or C

Let $X_i \sim N(0, \sigma_i^2)$ for $i = 1, 2, \dots, m_t$ and they are independent. Now we are to select K or C so that we can divide the set of observations in two groups. In the first cases if $\sigma_1 = \sigma_2 = \dots = \sigma_{m_t} = 1$, marginals of all X_i 's are the same distributions, and with constant K to select the highest K many we need to find a C such that X_i is selected if $|X_i| > C$. Let L_C denotes the number of X_i 's that are bigger than C and $E[L_C] = m_t P[|X_1| > C] = K$. Therefore $P[|X_1| > C] = \frac{K}{m_t}$ and if K is a random variable then $P[|X_1| > C] = E[\frac{K}{m_t}]$.

In the second case, we know that m_1 many σ_i^2 are $\sigma_0^2 + \tau^2$ and m_2 many are only σ_0^2 where $m_1 + m_2 = m_t$. To select K many X_i 's, we first have a C in mind. Among the K many X_i 's (Considering K constant), let K_1 many out of m_1 many and K_2 many out of m_2 many have come where $K_1 + K_2 = K$ with $|X_i| > C$. From now on we will use Y_i 's as our original observations and $X_i = \frac{Y_i}{\sigma_i}$. Hence:

$$E\left[\frac{K_1}{m_1}\right] = E\left(E\left[\frac{K_1}{m_1}\right] \middle| m_1\right) = P\left[|X_1| > \frac{C}{\sqrt{\sigma_0^2 + \tau^2}}\right] \quad (\text{Independent of } m_1) \quad (3)$$

$$E\left[\frac{K_2}{m_2}\right] = E\left(E\left[\frac{K_2}{m_2}\right] \middle| m_2\right) = P\left[|X_1| > \frac{C}{\sigma_0}\right] \quad (\text{Independent of } m_2) \quad (4)$$

$$K = E[K_1 + K_2] = m_1 P\left[|X_1| > \frac{C}{\sqrt{\sigma_0^2 + \tau^2}}\right] + m_2 P\left[|X_1| > \frac{C}{\sigma_0}\right] \quad (5)$$

Here $(\tau, \sigma_0, m_1, m_2)$ are known, so C can be found out. We may replace K by $E[K]$. So far we have not considered about the dispersion matrix Σ but with only the marginals. Now Σ will come into play if we consider power, level etc. K_2 is the number of false positives and $(m_1 - K_1)$ denote the number of false negatives.

$$\begin{aligned} E[m_1 - K_1] &= E[E[(m_1 - K_1)|m_1]] = E\left[m_1 E\left[1 - \frac{K_1}{m_1} \middle| m_1\right]\right] \\ &= E\left[m_1 P\left(|X_1| < \frac{C}{\sqrt{\sigma_0^2 + \tau^2}}\right)\right] = E[m_1] P\left[|X_1| < \frac{C}{\sqrt{\sigma_0^2 + \tau^2}}\right] \end{aligned} \quad (6)$$

$$E[K_2] = E[E[K_2|m_2]] = E[m_2] P\left[|X_1| > \frac{C}{\sigma_0}\right] \quad (7)$$

where $E[m_1] = m_t p_t$ and $E[m_2] = m_t(1 - p_t)$ and putting values from the previous equations we get the above.

In order to control the errors asymptotically we need large C i.e. $\frac{C}{\sigma_0}$ should be large and $\frac{C}{\sqrt{\sigma_0^2 + \tau^2}}$ should be small (τ being necessarily large). Now expression of expected risk will become the following :

$$R = \delta_0 m_t (1 - p_t) P\left[|X_1| > \frac{C}{\sigma_0}\right] + \delta_A m_t p_t P\left[|X_1| < \frac{C}{\sqrt{\sigma_0^2 + \tau^2}}\right] \quad (8)$$

There are some important points to be noted here. Given $(m_t, p_t, \delta_0, \delta_A, \sigma_0^2, \tau^2)$ we want to find a C such that the risk is minimised and call it C_{opt} . This quantity can be found out by simulation or by approximation of probabilities. If X_i 's were independent, the rejection region is $\bigcap_{i=1}^{m_t} \{|Y_i| > C\}$ i.e. the optimal rejection region. We can calculate C_{opt} in this case depending on the six parameters stated above. In the independent case C was a constant and did not depend on X_1, X_2, \dots, X_{m_t} [1] (Bogdan, Chakrabarty, Frommlet and Ghosh(2011)). If we force the same

rejection region here for the dependent case (with constant C) risk function as a function of C will not change. The important observation is that the risk function here depends only on the marginals. Therefore by this method similar to the case of independence, we can select the hypotheses to reject with the same risk function (which was computed in the independent case).

If we consider the correlated case instead of independent case stated above, we should expect the optimal risk to be less than that of the independent case. If the X_i 's are equi-correlated then $\bigcap_{i=1}^{m_t} \{|Y_i| > C(y_1, y_2, \dots, y_{m_t})\}$ where C is a symmetric function of $(y_1, y_2, \dots, y_{m_t})$ (If we consider permutation invariant optimal rejection region then we have to find the symmetric function C .)

As for example possible choices of C may be $\left(\frac{\sum_{i=1}^{m_t} |Y_i|^\beta}{m_t} \right)^{\frac{1}{\beta}}$ for positive β . [As in 3.3 Case III]

We have mentioned the expression of R i.e. the expected risk in terms of standard normal distributions and the corresponding losses, which can be approximated in the following way.

$$\begin{aligned} R &= \delta_0 m_t (1 - p_t) \frac{\sigma_0 \sqrt{2}}{C \sqrt{\pi}} e^{-\frac{C^2}{2\sigma_0^2}} + \delta_A m_t p_t \frac{C \sqrt{2}}{\sqrt{\pi(\sigma_0^2 + \tau^2)}} \\ &= \frac{V}{C} e^{-\frac{C^2}{2\sigma_0^2}} + UC \\ &= f(C) \end{aligned} \tag{9}$$

where $V = \delta_0 m_t (1 - p_t) \sigma_0 \sqrt{\frac{2}{\pi}}$ and $U = \frac{\delta_A m_t p_t}{\sqrt{(\sigma_0^2 + \tau^2)}} \sqrt{\frac{2}{\pi}}$. The above approximation is obtained by using two facts: for one part $\frac{C}{\sigma_0}$ large and for that we have used Mill's Ratio to get the approximation, and for the other part we have $\frac{C}{\sqrt{(\sigma_0^2 + \tau^2)}}$ small and simple approximation of standard normal density at 0. For the t_{21} part we have used the fact that for small x , $P[|N(0, 1)| < x] \approx 2x\phi(0)$. Now while approximating the other part i.e. t_{11} we have used the Mill's Ratio which is the following:

$$2 \left(\frac{1}{x} - \frac{1}{x^3} \right) \phi(x) < P[|N(0, 1)| > x] < \frac{2}{x} \phi(x)$$

Here $f(C)$ as defined above is a convex function of C for $U, V, a > 0$ as

$$\begin{aligned} f'(C) &= U - \frac{V}{C^2} e^{-aC^2} - 2aV e^{-aC^2} \\ &= U - V e^{-aC^2} \left(\frac{1}{C^2} + 2a \right) \end{aligned}$$

Where

$$\begin{aligned} U &= \frac{2\delta_A m p}{\sqrt{2\pi(\sigma_0^2 + \tau^2)}} \\ V &= \sigma_0 \delta_0 m (1 - p) \sqrt{\frac{2}{\pi}} \\ a &= \frac{1}{2\sigma_0^2} \end{aligned}$$

From the above calculation $f'(C)$ is an increasing of C which in turn implies the convexity of f . If C is a random variable and it is independent of X_1 then we have risk function $Ef(C)$ and :

$$E(f(C)) \geq f(E(C)) \quad (\text{by Jensen's Inequality}) \tag{10}$$

Here we have $(Y_1, Y_2, \dots, Y_{m_t}) = \underline{Y}$.

$$\underline{Y} | \underline{y} = \underline{v}_0 \sim N(0, D_{v_0} \Sigma_2 D_{v_0})$$

$$\sigma_i^2 = \text{Var}(Y_i) = \begin{cases} \sigma_0^2 + \tau^2 & \text{if } \nu_i = 1 \\ \sigma_0^2 & \text{if } \nu_i = 0 \end{cases}$$

Let $\text{Var}(Y_i) = \sigma_i^2$ (say), and we have defined $X_i = \frac{Y_i}{\sigma_i}$. We are to select the subset \mathcal{S} such that $i \in \mathcal{S}$ if and only if $\nu_{0i} = 1$. As we have discussed if Σ_2 is equi-correlated then optimal selection will be $i \in \mathcal{S}$ if and only if $Y_i^2 > C^*(\underline{Y})$. Here optimality is considered for the case $\delta_0 = \delta_A = 1$ and all permutation invariant (i.e. symmetric) selection procedure. We have the following results regarding the form of $C^*(\underline{Y})$.

Result 2 $C^*(\underline{Y})$ is a symmetric positive definite second degree polynomial in \underline{Y} in the following form:

$$C^*(\underline{Y}) = \left[a \sum_{i=1}^{m_t} Y_i^2 - m_t b \bar{Y}^2 \right] \quad \text{where } a > b \geq 0$$

Result 3 There exists a symmetric positive definite second degree polynomial $C_2^*(\underline{Y})$ such that $C^*(\underline{Y}) - C_2^*(\underline{Y})$ is of $\mathcal{O}(1)$, and $\frac{C^*(\underline{Y})}{m_t} \xrightarrow{\text{a.s.}} a_1$ as $m_t \rightarrow \infty$ and $a_1 > 0$ is a constant. Also $C_2^*(\underline{Y})$ is independent of Y_1 .

Proof of the above results will be stated in appendix part of this paper. In view of the two results stated just above, if we use $C_2^*(\underline{Y})$ in place of $C^*(\underline{Y})$ the procedure still remains asymptotically optimal.

Let \mathcal{S}_2 be given by $i \in \mathcal{S}_2$ if and only if $Y_i^2 > C_2^*(\underline{Y})$. In view of (10), with independence of Y_1 and $C_2^*(\underline{Y})$ and with the asymptotically optimal procedure, if we replace $C_2^*(\underline{Y})$ by $E[C_2^*(\underline{Y})]$ the risk will be improved. Therefore we can consider a better risk if we view $C_2^*(\underline{Y})$ as a constant. Thus we have asymptotically optimal selection procedure \mathcal{S}^* given by $i \in \mathcal{S}^*$ iff $Y_i^2 > C_0$ for some constant C_0 , where we have taken equi-correlated Σ_2 and symmetric i.e. permutation invariant selection procedures.

We have defined $f'(C)$ earlier. Considering the following in $f'(C)$

$$(2a + \frac{1}{C^2}) \approx 2a \quad (\text{As } C \text{ is large})$$

we get the maximiser C given by the following:

$$U = 2aVe^{-aC^2}$$

$$C = \sqrt{\frac{1}{a} \log \left(\frac{2aV}{U} \right)}$$

Putting the values:

$$C = \sqrt{2\sigma_0^2 \log \left(\frac{\delta_0(1-p)}{\delta_{AP}} \sqrt{\left(1 + \frac{\tau^2}{\sigma_0^2} \right)} \right)} \quad (11)$$

Thus we have proved

Theorem 1 If \mathcal{S}^* denotes set of hypotheses selected to be true, for the observations Y_1, Y_2, \dots, Y_{m_t} with equi-correlated set up, then the selection is asymptotically optimal (under $\delta_0 = \delta_A = 1$) if $i \in \mathcal{S}^* \Leftrightarrow |Y_i| > C$, $i = 1, 2, \dots, m_t$ where C is a fixed constant. An approximate optimal choice of C is given by (11).

5 Simulation:

Before getting in to the proposed procedures and analysis of the output of the results obtained we define a few quantities as follows:

$$T_1 = \left(\frac{\sum_{i=1}^{m_t} X_i^4}{n} \right)^{\frac{1}{4}}$$

$$T_2 = \left(\frac{\sum_{i=1}^{m_t} X_i^2}{n} \right)^{\frac{1}{2}}$$

$$T_3 = \left(\frac{\sum_{i=1}^{m_t} |X_i|}{n} \right)$$

5.1 Proposed Methods:

Now in order to determine the C that classifies the data into two groups to identify the signals we have derived the value of C under the normality set up in the last section. In this problem we have selected C in various ways and compared them in order to report the best case. The possible algorithms to select C are as follows:

- First method is what we have described above in case I, i.e. to select C using a Poisson random variable which is a very bad choice as seen by simulations. It performs bad with extremely high error rate.
- In the next case, we have used the fact that C is a symmetric function of Y_1, Y_2, \dots, Y_n . Therefore we have chosen simple symmetric functions as a choice of C for classifying the data into two sets and assign 1 to corresponding v_i 's with larger deviation and 0 otherwise. So we have chosen T_1, T_2 and T_3 .
- Another case is that we have simply classified the data on the basis of C determined by the above expression (11).
- Now we have proposed another iterative algorithm for determining the classifier C, which works good with reasonable error of false positive and false negative. To find the C we do the iterative steps in the following way:
 1. Start with $Z_0 = T_1$. Classify the vector of co-ordinate wise absolute value of X with this classifier.
 2. Now the co-ordinates of \underline{X} for which the corresponding absolute values are less than Z_0 and those which are greater than Z_0 form two groups of absolute values. Call the group means A_1 and A_2 respectively and obtain $Z_1 = \frac{A_1 + A_2}{2}$.
 3. Now go to step 1 with Z_1 and obtain Z_2, Z_3, \dots respectively.
 4. Terminate the process in the i-th step if $|Z_{i+1} - Z_i| < f$, where f is a predetermined very small cut of value.

This is because it can be shown that the sequence $\{Z_n\}_{n \geq 0}$ converges in general.

In the above points we have stated our possible choices of C as a classifier to solve the problem in presence of correlation. In the last point that we get a reasonable C with iterative limit of Z_i 's, is justified by the following result.

Result 4 Let w_1, w_2, \dots, w_m be m positive observations. Then within group variance:

$$\sum_{w_i \leq C} (w_i - \bar{w}_1)^2 + \sum_{w_i > C} (w_i - \bar{w}_2)^2$$

is minimised for a value of C which necessarily satisfies $C = \frac{\bar{w}_1 + \bar{w}_2}{2}$ with \bar{w}_i giving the mean of the i -th group.

5.2 Discussion

As we have stated earlier, our problem now boils down to finding a suitable C which will classify the observations coming from an underlying set up. We did not have any real standard data to test run our process, so we have simulated and performed the tests to check the validity of our method and compared the cases having non-zero correlation terms with the independent cases. As we have developed our method, we have shown that our process with non-zero correlation coefficient in the equi-correlated set up is at least as good as the independent case in terms of the risk function.

While devising the methods for our problem, we needed an ideal choice of C , so that we could compare our methods with the ideal case and get an estimate of efficiency. Now the problem lies in selection of the ideal C i.e. the best performing C while classifying the simulated data. In order to choose that particular C we applied brute force to approximately find the closest ideal C to work with. As we have seen that we have to classify according to a measure of variance, we have taken modulus value of the simulated X vector co-ordinatewise. Now we looked for the range of the absolute values of the co-ordinates of X and started with the lowest point of the range as our starting C . After classification with this C and finding out the error (Sum of false positive and false negative), we fixed an increment value for C . Every time we increase C with the corresponding fixed increment value, classify the simulated data (from a pre-decided fixed set up, i.e. fixing $m_t, \beta, \sigma_0, \tau$, and ρ) and find out the total error in the case. In this way we can get a sequence of total errors for different values of C . Among them we select that C which corresponds to the minimum total error among these sequence of errors and call it our ideal C and report the corresponding total error as the ideal case total error.

Now as we have described the process of selecting the best C and determined the ideal case total error by brute force, there are some subtle points which needs to be understood properly in order to justify the process. As the process goes on finding, there may be multiple choices of best C , which provides the same total error at the end. But we do not need to bother about those multiple C 's in this case at all, because our purpose for generating this ideal case is to get the idea of the total error in the best possible choice of C and create an ideal case to compare with for testing our proposed methods of choice of C . So we select any one of these best C 's as our C and classify the simulated data to get the total error and repeat this process 1000 times to get the expected error in the ideal case.

In the following paragraph, we will describe the observations obtained directly from the simulation studies where the correlation coefficients are equal and non-negative. As we have stated earlier that C is a symmetric function of X_1, X_2, \dots, X_n so we have selected some typical functions to know about the behaviour of the total error. From the simulation studies, we have some good observations to make about our procedure. We claimed that our procedure is at least as good as in the case of independent set up in terms of risk function. This phenomena is reflected in the simulation studies as well. If we go through the table we can easily see that irrespective of the method of choosing the C , our claim holds. Apart from that as the correlation gets high, this methods perform even better.

There is another observation that can be made from the simulated data sets. In this process τ is generally assumed to be larger than σ_0 , which in turn helps the process of classification in this way to work properly. This is because we are dividing the co-ordinates of the simulated random variable into two parts according to a measure of variance and estimating the corresponding v_i to be 1 having larger variance and 0 to the rest of them. Now as we can see from the simulation studies that if the ratio $\frac{\tau}{\sigma_0}$ is large then the classification is good and the expected number of both false positive and false negative decrease which is expected. The T_3 performs better compared to T_1 and T_2 if the ratio is high.

In case of C determined by the expression (11) stated above, we can see that the expression depends on δ_0 and δ_A . In our set up we have assumed that these two values are same and equal to 1 and carried out the simulation procedure with C as in expression (11). From the simulation table of total error presented here, we can see that this choice of C works better if β is higher i.e. it is sensitive to β and it gets extremely close to the ideal case for higher values of β . In all the cases, performance of T_1 and T_2 are more or less similar to each other. The algorithm performs better than the two expressions T_1 and T_2 , when the ratio $\frac{\tau}{\sigma_0}$ is not comparatively low. But in the over all set up the algorithm works better than the two expressions mentioned as m_t and β becomes larger. Another interesting observation is that for T_1, T_2 and for the choice of C with the algorithm stated above, we can see that for smaller value of β the bias is toward the false negative values i.e. expected false negative is almost uniformly larger than the expected false positives. But in case of higher values of β the bias gets reversed. In case of C determined by the formula derived above (in the last section) the bias is almost always toward the false negative cases.

If we look at the cases with equal and negative correlation coefficient, we can easily see that the observations made above, holds easily. So our method works also for negative ρ as well and the performance is at least as good as the independent case in terms of the risk functions. As m_t increases all the possible choices of C works better gradually in this set up also.

Now to compare the performance of the methods we calculated the discrepancies of various methods using the following formula:

$$\text{Discrepancy in Percentage} = 100 \times \frac{E_K - E_{K_0}}{E_K}$$

where E_K is the total error in the corresponding choice of C and E_{K_0} is the total error in the ideal choice of C . Here by total error we mean the sum of the expected number of false positive and expected number of false negative cases.

Remark 3 Note that, we have made classification in the ideal case with the knowledge of which observation comes from which σ_i^2 . Here in practice, it may not be achievable as the classifier is not a function of $Y_1^2, Y_2^2, \dots, Y_{m_t}^2$ alone. Thus the ideal case can be looked upon as some lower bound which may not be achievable even in the limit.

6 Appendix

1. Proof of Equation (2):

$$\begin{aligned} R(\underset{\sim}{v}, \underset{\sim}{v}^*) &= EE[L(\underset{\sim}{v}, \underset{\sim}{v}^*) | \underset{\sim}{v} = \underset{\sim}{v}_0] \\ &= \sum_{\underset{\sim}{v}_0} p_{\underset{\sim}{v}_0} E \left[\sum_{i=1}^m \delta_0 \mathbb{1}_{[(v_{0i} - v_i^*)=1]} + \sum_{i=1}^m \delta_A \mathbb{1}_{[(v_{0i} - v_i^*)=-1]} \right] \\ &= \sum_{i=1}^m E \left[\sum_{\underset{\sim}{v}_0} p_{\underset{\sim}{v}_0} \left(\delta_0 \mathbb{1}_{[(v_{0i} - v_i^*)=1]} + \delta_A \mathbb{1}_{[(v_{0i} - v_i^*)=-1]} \right) \right] \\ &= \sum_{i=1}^m \left[\sum_{\underset{\sim}{v}_0} p_{\underset{\sim}{v}_0} \left(\delta_0 E(\mathbb{1}_{(v_{0i}=1)} | v_i^* = 0) P(v_i^* = 0) + \delta_A E(\mathbb{1}_{(v_{0i}=0)} | v_i^* = 1) P(v_i^* = 1) \right) \right] \\ &= \sum_{i=1}^m \delta_0 (1-p) t_{1i} + \delta_A p t_{2i} \end{aligned}$$

2. Proof of Lemma(1):

(a) As we have described earlier

$$\underline{X} \sim \sum_{v_0} p_{v_0} N(0, D_{v_0} \Sigma_2 D_{v_0})$$

We need to select i_1, i_2, \dots, i_k for which v_{i_j} 's are 1 for $j = 1, 2, \dots, k$. First we consider marginals of X_i^2 and X_j^2 with their marginal variance σ_i^2 and σ_j^2 as their variances respectively. Then

$$X_i^2 \sim \sigma_i^2 \chi_{(\frac{1}{2}, \frac{1}{2})}^2$$

From this it easily follows that $X_i^2 \leq_{st} X_j^2$ if and only if $\sigma_i^2 \leq \sigma_j^2$.

(b) Here we shall prove $X_i^2|Z \leq_{st} X_j^2|Z$ when the inequality in variance holds as stated above. Here Z is a subset of $\{X_1, X_2, \dots, X_m\}$ deleted by X_i and X_j respectively. Now in the equi-correlated set up without loss of generality, as we can assume that instead of $i \neq j$ we may simply work with 1 and 2. Define

$$U = \left[\left(\frac{X_1}{\sigma_1}, \frac{X_2}{\sigma_2} \right) \middle| \left(\frac{X_3}{\sigma_3}, \frac{X_4}{\sigma_4} \dots \frac{X_m}{\sigma_m} \right) \right]$$

This quantity is free of σ_i^2 and $\left(\frac{X_1}{\sigma_1}, \frac{X_2}{\sigma_2}, \frac{X_3}{\sigma_3}, \frac{X_4}{\sigma_4} \dots \frac{X_m}{\sigma_m} \right)$ has exchangeable distributions. Now $U = (U_1, U_2)$ which are exchangeable. $(X_1, X_2) = (\sigma_1 U_1, \sigma_2 U_2)$ which has equi-correlated matrix \mathcal{R}^* . Hence by part (a) the result follows.

3. Proof of Result 1:

The proof of the result follows from the fact that the Lorentz curve of Z at abscissa

$$p = \int_0^{\bar{Z}} dF_Z(t)$$

is parallel to the line joining $(0, 0)$ and $(1, 1)$.

4. Proof of Result 2:

Let us define $v_i^{(\mathcal{S})} = 1$ iff $i \in \mathcal{S}$. Now the subset \mathcal{S} is said to be optimal if and only if the following inequality holds:

$$\log \left[\frac{f_{v^{(\mathcal{S})}}(\underline{y})}{f_{\underline{v}}(\underline{y})} \right] \geq 0 \quad \left(\forall \quad \underline{v}^{(\mathcal{S})} \neq \underline{v} \right) \quad (12)$$

In (12), $\log \left[\frac{f_{v^{(\mathcal{S})}}(\underline{y})}{f_{\underline{v}}(\underline{y})} \right]$ is a polynomial of second degree $\forall \underline{y}$ and the required region for \mathcal{S} is obtained by intersection of the regions, derived from the polynomial. We know, $C^*(\underline{Y})$ will be symmetric positive definite and homogeneous (from the permutation invariance and the set up of the region $Y_i^2 > C^*(\underline{Y})$). From the above paragraph we observe that $C^*(\underline{Y})$ is also of second degree. Hence $C^*(\underline{Y})$ which is a symmetric, p.d., quadratic form implies that it has the form mentioned in the statement, proving Result(2).

5. Proof of Result 3:

Let us define

$$C_1^*(\underline{Y}) = \left[a \sum_{i=2}^{m_t} Y_i^2 - m_t b \bar{Y}_1^2 \right] \quad \left(\text{where } \bar{Y}_1 = \frac{\sum_{i=2}^{m_t} Y_i^2}{m_t} \right)$$

Now we note the following:

$$\begin{aligned}
C^*(\underline{Y}) - C_1^*(\underline{Y}) &= aY_1^2 - m_t b(\bar{Y}^2 - \bar{Y}_1^2) \\
&= aY_1^2 - b.Y_1.(\bar{Y} + \bar{Y}_1) \\
&\approx \mathcal{O}(1)
\end{aligned}$$

$C_1^*(\underline{Y})$ is not independent of Y_1 , as \exists correlation between Y_i and $Y_j \forall i \neq j$.

Now we define $C_2^*(\underline{Y})$ as follows:

$$C_2^*(\underline{Y}) = C_1^*(\underline{Y} - \underline{\sigma}.\bar{X}_1) \quad \left(\text{where } \underline{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_{m_t})^T \right)$$

Now here in the above equation

$$\begin{aligned}
\underline{\sigma}.\bar{X}_1 &= (\sigma_1\bar{X}_1, \sigma_2\bar{X}_1, \dots, \sigma_{m_t}\bar{X}_1) \\
\sigma_i &= \text{Var}(Y_i) \\
\bar{X}_1 &= \frac{\sum_{i=2}^{m_t} X_i}{m_t - 1} = \frac{\sum_{i=2}^{m_t} \frac{Y_i}{\sigma_i}}{m_t - 1}
\end{aligned}$$

$C_1^*(\underline{Y} - \underline{\sigma}.\bar{X}_1)$ is independent with Y_1 because $(\underline{Y} - \underline{\sigma}.\bar{X}_1)$ follows multivariate normal distribution and as the following holds:

$$\text{Cov}(Y_2, X_1) = \rho\sigma_2 \quad \text{Cov}(X_1, \sigma_2\bar{X}_1) = \rho\sigma_2 \quad \text{Cov}(X_1, \underline{Y}^* - \underline{\sigma}.\bar{X}_1) = 0$$

Now it is easy to see that $\frac{C^*(\underline{Y})}{m_t} \xrightarrow{\text{a.s.}} a_1$, as $\bar{Y}^2 \xrightarrow{\text{a.s.}} 0$ and the following occurs:

$$\begin{aligned}
C^*(\underline{Y}) - C_2^*(\underline{Y}) &= \left[a \sum_{i=1}^{m_t} Y_i^2 - m_t b \bar{Y}^2 \right] - C_1^*(\underline{Y} - \underline{\sigma}.\bar{X}_1) \\
&= \left[a \sum_{i=1}^{m_t} Y_i^2 - m_t b \bar{Y}^2 \right] - \left[a \sum_{i=2}^{m_t} (Y_i - \sigma_i \bar{X}_1)^2 - m_t b (\bar{Y}_1 - \bar{X}_1.\bar{\sigma})^2 \right] \quad \left(\text{where } \bar{\sigma} = \frac{\sum_{i=2}^{m_t} \sigma_i}{m_t} \right) \\
&= aY_1^2 + 2a\bar{X}_1 \sum_i \sigma_i Y_i - a\bar{X}_1^2 \sum_{i=2}^{m_t} \sigma_i^2 + \mathcal{O}(1) \\
&\approx \mathcal{O}(1)
\end{aligned}$$

6. Proof of Result 4:

Let us assume a continuous p.d.f. of w and call it f_w . Then within group variance $V_w(C)$ of the two groups obtained from w , using C , is a continuous function of C . We consider

$$\frac{\partial V_w(C)}{\partial C} = 0$$

and obtain the result. Now as the result holds for continuous p.d.f., it is easy to see that it holds for the discrete case also.

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Table 1: Comparison of Total Error with Non-negative Rho

m_t	β	σ_0	τ	ρ	T_1	T_2	T_3	Algorithm	Determined	ideal
80	0.3	1	15	0	11.053	11.143	15.46	12.052	11.966	9.473
80	0.3	1	15	0.1	11.143	11.009	15.144	11.782	12.066	9.519
80	0.3	1	15	0.7	8.338	8.31	11.337	9.292	11.945	6.959
80	0.3	1	90	0	8.534	8.423	5.531	10.453	6.362	4.356
80	0.3	1	90	0.1	8.581	8.544	5.65	10.317	6.303	4.396
80	0.3	1	90	0.7	6.369	6.888	3.966	7.73	6.325	2.943
80	0.3	3	15	0	18.098	17.861	23.348	17.398	20.324	13.912
80	0.3	3	15	0.1	17.963	17.865	23.154	17.244	20.22	13.627
80	0.3	3	15	0.7	14.825	14.482	20.295	14.002	19.935	11.403
80	0.3	3	90	0	9.189	9.176	10.35	11.181	14.694	7.072
80	0.3	3	90	0.1	8.996	9.199	10.586	10.99	14.825	7.025
80	0.3	3	90	0.7	7.099	7.051	7.28	8.671	14.814	4.987
80	0.7	1	15	0	17.432	17.526	28.482	15.539	2.319	2.019
80	0.7	1	15	0.1	16.782	17.369	28.892	15.516	2.406	2.01
80	0.7	1	15	0.7	16.381	16.467	29.156	15.636	2.49	1.533
80	0.7	1	90	0	6.823	6.798	21.241	5.453	1.252	0.896
80	0.7	1	90	0.1	6.78	6.717	21.271	5.426	1.295	0.957
80	0.7	1	90	0.7	5.226	5.266	19.847	4.838	1.253	0.587
80	0.7	3	15	0	22.313	21.969	31.078	23.244	3.7	2.972
80	0.7	3	15	0.1	22.303	22.331	31.319	23.388	3.651	3.144
80	0.7	3	15	0.7	23.163	23.784	32.407	25.952	3.7	2.461
80	0.7	3	90	0	13.402	13.048	26.318	10.718	2.871	1.447
80	0.7	3	90	0.1	12.452	13.289	26.422	10.907	2.864	1.438
80	0.7	3	90	0.7	11.568	11.91	25.648	9.408	2.861	0.976
180	0.3	1	15	0	21.422	21.275	37.473	21.383	21.861	18.091
180	0.3	1	15	0.1	21.263	21.293	37.901	20.934	21.636	18.037
180	0.3	1	15	0.7	15.361	15.511	29.418	16.91	21.62	13.248
180	0.3	1	90	0	13.589	13.737	11.576	18.931	11.483	8.526
180	0.3	1	90	0.1	13.458	13.281	11.504	18.742	11.279	8.553
180	0.3	1	90	0.7	10.264	10.302	7.356	14.584	11.214	5.812
180	0.3	3	15	0	38.683	38.651	55.029	35.032	36.397	26.14
180	0.3	3	15	0.1	38.9	38.665	55.11	35.059	36.055	26.035
180	0.3	3	15	0.7	32.719	33.522	48.907	27.964	36.1	21.611
180	0.3	3	90	0	15.637	15.497	25.676	19.784	26.489	13.863
180	0.3	3	90	0.1	15.287	15.353	26.02	19.741	26.946	13.742
180	0.3	3	90	0.7	11.654	11.442	17.841	15.223	27.093	9.592
180	0.7	1	15	0	44.007	44.126	69.14	43.673	3.078	2.679
180	0.7	1	15	0.1	44.311	44.158	69.31	43.176	3.201	2.697
180	0.7	1	15	0.7	46.1	45.9	71.579	45.311	3.134	2.018
180	0.7	1	90	0	19.78	19.126	59.069	12.498	1.666	1.211
180	0.7	1	90	0.1	19.573	19.773	58.688	13.925	1.625	1.184
180	0.7	1	90	0.7	15.559	15.972	57.509	12.136	1.499	0.821
180	0.7	3	15	0	52.69	52.419	72.593	54.388	4.565	3.835
180	0.7	3	15	0.1	52.683	52.583	73.073	54.946	4.688	3.867
180	0.7	3	15	0.7	57.25	57.123	77.13	62.967	4.592	3.199
180	0.7	3	90	0	35.13	35.145	65.981	29.8	3.659	1.923
180	0.7	3	90	0.1	36.024	35.697	66.173	28.454	3.6	1.944
180	0.7	3	90	0.7	35.567	34.178	67.827	26.402	3.779	1.392

Table 2: Comparison of Total Error with Negative Rho

m_t	β	σ_0	τ	ρ	T_1	T_2	T_3	Algorithm	Determined	ideal
80	0.3	1	15	-0.00633	11.062	11.045	15.178	11.929	11.97	9.631
80	0.3	1	90	-0.00633	8.55	8.43	5.589	10.334	6.415	4.445
80	0.3	3	15	-0.00633	18.085	17.771	23.237	17.452	20.293	13.735
80	0.3	3	90	-0.00633	9.062	9.195	10.391	11.041	14.765	7.207
80	0.7	1	15	-0.00633	16.894	17.003	28.578	15.768	2.385	2.062
80	0.7	1	90	-0.00633	6.492	6.844	21.449	5.885	1.234	0.943
80	0.7	3	15	-0.00633	21.952	22.174	31.259	23.129	3.696	3.079
80	0.7	3	90	-0.00633	12.603	13.307	26.598	10.989	2.807	1.466
180	0.3	1	15	-0.00279	21.128	21.577	37.549	21.427	21.72	18.201
180	0.3	1	90	-0.00279	13.346	13.499	11.577	18.955	11.246	8.532
180	0.3	3	15	-0.00279	38.623	39.045	54.933	34.587	35.979	26.261
180	0.3	3	90	-0.00279	15.471	15.58	25.512	19.962	26.583	13.562
180	0.7	1	15	-0.00279	43.747	44.092	69.131	42.347	3.186	2.663
180	0.7	1	90	-0.00279	19.078	20.115	58.197	13.163	1.564	1.199
180	0.7	3	15	-0.00279	52.558	51.953	72.996	54.007	4.831	4.028
180	0.7	3	90	-0.00279	36.022	35.155	65.59	29.725	3.754	1.964

Table 3: Comparison of Discrepancies (in Percentage) of Various Methods for Negative Rho

m_t	β	σ_0	τ	ρ	T_1	T_2	T_3	Algorithm	Determined
80	0.3	1	15	-0.00633	12.936	12.802	36.546	19.264	19.541
80	0.3	1	90	-0.00633	48.012	47.272	20.469	56.987	30.709
80	0.3	3	15	-0.00633	24.053	22.711	40.892	21.298	32.317
80	0.3	3	90	-0.00633	20.470	21.620	30.642	34.725	51.189
80	0.7	1	15	-0.00633	87.794	87.873	92.785	86.923	13.543
80	0.7	1	90	-0.00633	85.474	86.222	95.604	83.976	23.582
80	0.7	3	15	-0.00633	85.974	86.114	90.150	86.688	16.694
80	0.7	3	90	-0.00633	88.368	88.983	94.488	86.659	47.773
180	0.3	1	15	-0.00279	13.854	15.646	51.527	15.056	16.202
180	0.3	1	90	-0.00279	36.071	36.795	26.302	54.988	24.133
180	0.3	3	15	-0.00279	32.007	32.742	52.194	24.073	27.010
180	0.3	3	90	-0.00279	12.339	12.953	46.841	32.061	48.982
180	0.7	1	15	-0.00279	93.913	93.960	96.148	93.711	16.416
180	0.7	1	90	-0.00279	93.715	94.039	97.940	90.891	23.338
180	0.7	3	15	-0.00279	92.336	92.247	94.482	92.542	16.622
180	0.7	3	90	-0.00279	94.548	94.413	97.006	93.393	47.682

Table 4: Comparison of Discrepancies (in Percentage) of Various Methods for Non-negative Rho

m_t	β	σ_0	τ	ρ	T_1	T_2	T_3	Algorithm	Determined
80	0.3	1	15	0	14.295	14.987	38.726	21.399	20.834
80	0.3	1	15	0.1	14.574	13.534	37.143	19.207	21.109
80	0.3	1	15	0.7	16.539	16.258	38.617	25.108	41.741
80	0.3	1	90	0	48.957	48.284	21.244	58.328	31.531
80	0.3	1	90	0.1	48.771	48.549	22.195	57.391	30.255
80	0.3	1	90	0.7	53.792	57.274	25.794	61.928	53.470
80	0.3	3	15	0	23.130	22.110	40.415	20.037	31.549
80	0.3	3	15	0.1	24.139	23.722	41.146	20.975	32.606
80	0.3	3	15	0.7	23.083	21.261	43.814	18.562	42.799
80	0.3	3	90	0	23.038	22.929	31.671	36.750	51.872
80	0.3	3	90	0.1	21.910	23.633	33.639	36.078	52.614
80	0.3	3	90	0.7	29.751	29.272	31.497	42.486	66.336
80	0.7	1	15	0	88.418	88.480	92.911	87.007	12.937
80	0.7	1	15	0.1	88.023	88.428	93.043	87.046	16.459
80	0.7	1	15	0.7	90.642	90.690	94.742	90.196	38.434
80	0.7	1	90	0	86.868	86.820	95.782	83.569	28.435
80	0.7	1	90	0.1	85.885	85.753	95.501	82.363	26.100
80	0.7	1	90	0.7	88.768	88.853	97.042	87.867	53.152
80	0.7	3	15	0	86.680	86.472	90.437	87.214	19.676
80	0.7	3	15	0.1	85.903	85.921	89.961	86.557	13.887
80	0.7	3	15	0.7	89.375	89.653	92.406	90.517	33.486
80	0.7	3	90	0	89.203	88.910	94.502	86.499	49.599
80	0.7	3	90	0.1	88.452	89.179	94.558	86.816	49.791
80	0.7	3	90	0.7	91.563	91.805	96.195	89.626	65.886
180	0.3	1	15	0	15.549	14.966	51.723	15.395	17.245
180	0.3	1	15	0.1	15.172	15.291	52.410	13.839	16.634
180	0.3	1	15	0.7	13.756	14.590	54.966	21.656	38.723
180	0.3	1	90	0	37.258	37.934	26.348	54.963	25.751
180	0.3	1	90	0.1	36.447	35.600	25.652	54.365	24.169
180	0.3	1	90	0.7	43.375	43.584	20.990	60.148	48.172
180	0.3	3	15	0	32.425	32.369	52.498	25.383	28.181
180	0.3	3	15	0.1	33.072	32.665	52.758	25.739	27.791
180	0.3	3	15	0.7	33.950	35.532	55.812	22.718	40.136
180	0.3	3	90	0	11.345	10.544	46.008	29.928	47.665
180	0.3	3	90	0.1	10.107	10.493	47.187	30.389	49.002
180	0.3	3	90	0.7	17.693	16.169	46.236	36.990	64.596
180	0.7	1	15	0	93.912	93.929	96.125	93.866	12.963
180	0.7	1	15	0.1	93.913	93.892	96.109	93.753	15.745
180	0.7	1	15	0.7	95.623	95.603	97.181	95.546	35.609
180	0.7	1	90	0	93.878	93.668	97.950	90.310	27.311
180	0.7	1	90	0.1	93.951	94.012	97.983	91.497	27.138
180	0.7	1	90	0.7	94.723	94.860	98.572	93.235	45.230
180	0.7	3	15	0	92.722	92.684	94.717	92.949	15.991
180	0.7	3	15	0.1	92.660	92.646	94.708	92.962	17.513
180	0.7	3	15	0.7	94.412	94.400	95.852	94.920	30.335
180	0.7	3	90	0	94.526	94.528	97.086	93.547	47.445
180	0.7	3	90	0.1	94.604	94.554	97.062	93.168	46.000
180	0.7	3	90	0.7	96.086	95.927	97.948	94.728	63.165